Notes on Varieties and Commutative Algebra

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November 2024

Contents

1 Fundamental Definitions

Before diving into the main results, we establish some fundamental definitions that will be used throughout this work.[1]

[1] This expository paper was written after a summer MAT599 course with Dr. Bill Chin (DePaul) on Commutative Algebra and Algebraic Geometry. Additionally, thanks to Dr. Nicholas Ramsey (DePaul) for informally mentoring me during the Fall research assistantship. Lastly, thanks to my friend Drew Melman-Rogers (UChicago) for his thoughtful comments on the original draft and for guiding me this summer.

Definition 1.1 (Noetherian Ring)**.** A commutative ring R is called Noetherian if every ideal of R is finitely generated. Equivalently, R is Noetherian if and only if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

eventually stabilizes, meaning there exists N such that $I_n = I_N$ for all $n \geq N$.

Definition 1.2 (Regular Map)**.** Let V and W be varieties over a field K. A regular map (or morphism) $\phi : V \to W$ is a function that can be represented locally by rational functions with denominators that don't vanish. More precisely, for each point $p \in V$, there exists an open neighborhood U containing p and rational functions $f_1, \ldots, f_n \in K(V)$ that are defined at all points of U such that

$$\phi|_U(x) = (f_1(x), \ldots, f_n(x))$$

for all $x \in U$.

Definition 1.3 (Polynomial Map)**.** Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine varieties. A polynomial map $\phi : V \to W$ is a function of the form

$$\phi(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

where each $f_i \in K[x_1, \ldots, x_n]$, such that $\phi(V) \subseteq W$. Every polynomial map is regular, but not every regular map is polynomial.

2 Varieties and Algebraic Sets

Definition 2.1 (Affine Algebraic Set)**.** Let K be an algebraically closed field. An algebraic set in $\mathbb{A}^n_K$ is a subset $V \subseteq \mathbb{A}^n_K$ of the form

$$V = V(I) = \{x \in \mathbb{A}^n_K : f(x) = 0 \text{ for all } f \in I\}$$

where I is an ideal in $K[x_1, \ldots, x_n]$.

Definition 2.2 (Affine Variety)**.** An affine variety is an irreducible algebraic set, i.e., an algebraic set that cannot be written as the union of two proper algebraic subsets.

3 Hilbert's Basis Theorem

Theorem 3.1 (Hilbert's Basis Theorem)**.** *If R is a Noetherian ring, then $R[X]$ is Noetherian. Consequently, if K is a field, then $K[X_1, \ldots, X_n]$ is Noetherian.*

Proof. Let I be an ideal in $R[X]$. For $f \in R[X]$, we denote by $\mathrm{lt}(f)$ the leading term of f and by $\mathrm{lc}(f)$ the leading coefficient of f.

For any ideal $I \subseteq R[X]$ and $d \geq 0$, define:

$$L_d(I) = \{0\} \cup \{\mathrm{lc}(f) : f \in I \text{ with } \deg(f) = d\}$$

Lemma 3.2 (Ideal Property)**.** *For each $d \geq 0$, $L_d(I)$ is an ideal in R.*

Proof. Let $a, b \in L_d(I)$. If either is 0, clearly $a + b \in L_d(I)$. Otherwise, let $f, g \in I$ with $\mathrm{lc}(f) = a$ and $\mathrm{lc}(g) = b$. Consider $h = f + g$. Either $\deg(h) < d$ or $\mathrm{lc}(h) = a + b$, so $a + b \in L_d(I)$.

For $r \in R$ and $a \in L_d(I)$, if $a = 0$ then $ra = 0 \in L_d(I)$. Otherwise, let $f \in I$ with $\mathrm{lc}(f) = a$. Then rf has leading coefficient ra, so $ra \in L_d(I)$. □

Lemma 3.3 (Stabilization)**.** *The ascending chain of ideals*

$$L_0(I) \subseteq L_0(I) + L_1(I) \subseteq L_0(I) + L_1(I) + L_2(I) \subseteq \cdots$$

stabilizes, as R is Noetherian.

Since R is Noetherian, there exists $N \geq 0$ such that:

$$L_0(I) + \cdots + L_N(I) = L_0(I) + \cdots + L_N(I) + L_{N+k}(I)$$

for all $k \geq 0$.

For each $d \leq N$, since R is Noetherian, $L_d(I)$ is finitely generated. Let $\{a_{d1}, \ldots, a_{dr_d}\}$ generate $L_d(I)$. For each a_{di}, choose $f_{di} \in I$ with $\text{lc}(f_{di}) = a_{di}$ and $\deg(f_{di}) = d$.

Let F be the finite set of all chosen f_{di}, and let $J = \langle F \rangle$ be the ideal generated by F in $R[X]$.

Lemma 3.4 (Key Reduction)**.** *For any $g \in I$, there exists $g' \in I$ such that:*

- $g - g' \in J$
- *Either* $g' = 0$ *or* $\deg(g') < \deg(g)$

The proof follows by considering cases based on whether $\deg(g) \leq N$ or $\deg(g) > N$ and using the properties of the leading term ideals we constructed.

Finally, we can show $I = J$. Clearly $J \subseteq I$ as $F \subseteq I$. For the reverse inclusion, let $g \in I$. We use induction on $\deg(g)$. If $g = 0$, clearly $g \in J$. Otherwise, by the Key Reduction Lemma, find g' with $g - g' \in J$ and $\deg(g') < \deg(g)$. By induction, $g' \in J$, therefore $g = (g - g') + g' \in J$. □

4 Noether's Normalization Lemma

Theorem 4.1 (Noether's Normalization Lemma)**.** *Let A be a finitely generated K-algebra, where K is a field. Then there exist elements $y_1, \ldots, y_d \in A$ that are algebraically independent over K such that A is integral over $K[y_1, \ldots, y_d]$.*

Proof.

Lemma 4.2 (Reduction to Standard Form)**.** *It suffices to prove the theorem for $A = K[x_1, \ldots, x_n]/I$ where I is an ideal in $K[x_1, \ldots, x_n]$.*

Proof. Since A is finitely generated over K, we can write $A = K[a_1, \ldots, a_n]$ for some elements $a_1, \ldots, a_n \in A$. Define a surjective homomorphism $\phi : K[X_1, \ldots, X_n] \to A$ by $\phi(X_i) = a_i$. Let $I = \ker(\phi)$. Then $A \cong K[X_1, \ldots, X_n]/I$. □

Lemma 4.3 (Base Case)**.** *The theorem holds for $n = 1$.*

Proof. For $n = 1$, $A = K[x]/(f)$ for some polynomial f. If $f = 0$, then $A \cong K[x]$ and we take $d = 1$, $y_1 = x$. If $f \neq 0$, then A is finite-dimensional over K and we take $d = 0$. □

Lemma 4.4 (Polynomial Separation)**.** *Let $f \in K[X_1, \ldots, X_n]$ be non-zero. After a suitable linear change of variables, f becomes monic in X_n.*

Proof. Write $f = \sum_\alpha c_\alpha X^\alpha$ where α ranges over multi-indices. Consider the transformation:

$$X_i' = X_i + \lambda_i X_n \text{ for } i < n$$

$$X_n' = X_n$$

Under this transformation, the coefficient of the highest power of X_n is a non-zero polynomial in $\lambda_1, \ldots, \lambda_{n-1}$. Choose values making this coefficient non-zero, then divide by it. □

Now we proceed by induction on n. Assume the theorem holds for algebras generated by fewer than n elements.

If $x_1, \ldots, x_n$ are algebraically independent mod I, we're done. Otherwise, there exists a non-zero polynomial $f \in I$. By the Polynomial Separation Lemma, after a linear change of variables:

$$f = X_n^d + a_1 X_n^{d-1} + \cdots + a_d$$

where $a_i \in K[X_1, \ldots, X_{n-1}]$.

Define new variables:

$$y_i = X_i + X_n^i \text{ for } 1 \leq i \leq n-1$$

Lemma 4.5 (Key Property)**.** *The map $K[y_1, \ldots, y_{n-1}] \to K[X_1, \ldots, X_n]/I$ is injective.*

Proof. The transformation $(X_1, \ldots, X_{n-1}) \mapsto (y_1, \ldots, y_{n-1})$ is triangular with determinant 1. If $g(y_1, \ldots, y_{n-1}) = 0 \mod I$, substitute the expressions for y_i. The highest degree terms in X_n cannot cancel unless $g = 0$. □

Lemma 4.6 (Integrality)**.** *X_n is integral over $K[y_1, \ldots, y_{n-1}]$ modulo I.*

Proof. Recall that after our change of coordinates, $f \in I$ has the form

$$f = X_n^d + a_1 X_n^{d-1} + \cdots + a_d$$

where $a_i \in K[X_1, \ldots, X_{n-1}]$, and we defined

$$y_i = X_i + X_n^i \text{ for } 1 \leq i \leq n-1$$

Solving for each X_i gives:

$$X_i = y_i - X_n^i \text{ for } 1 \leq i \leq n-1$$

Now substitute these expressions into the coefficients a_i of f. Each a_i is a polynomial in the X_j's, so after substitution we get:

$$a_i = b_i(y_1, \ldots, y_{n-1}, X_n)$$

where each b_i is a polynomial in the y_j's and X_n.

Therefore, modulo I, X_n satisfies:

$$X_n^d + b_1(y_1, \ldots, y_{n-1}, X_n) X_n^{d-1} + \cdots + b_d(y_1, \ldots, y_{n-1}, X_n) = 0$$

Let D be larger than both d and the degree of X_n in any b_i. Group terms by powers of X_n up to degree D. The coefficient of each power X_n^k will be a polynomial in $y_1, \ldots, y_{n-1}$. Thus we obtain a monic polynomial equation for X_n with coefficients in $K[y_1, \ldots, y_{n-1}]$, showing that X_n is integral over $K[y_1, \ldots, y_{n-1}]$ modulo I. □

Completion of Proof of Noether's Normalization. Having established the integrality of X_n over $K[y_1, \ldots, y_{n-1}]$ modulo I, we complete the proof as follows:

1. By the Key Property proved earlier, $K[y_1, \ldots, y_{n-1}]$ embeds in $A = K[X_1, \ldots, X_n]/I$.
2. Apply the inductive hypothesis to $K[y_1, \ldots, y_{n-1}]$ (which has $n-1$ generators). This gives us algebraically independent elements $z_1, \ldots, z_d \in K[y_1, \ldots, y_{n-1}]$ such that $y_1, \ldots, y_{n-1}$ are integral over $K[z_1, \ldots, z_d]$.
3. Now we show each X_i is integral over $K[y_1, \ldots, y_{n-1}]$:
   - For each $i < n$, we have $X_i = y_i - X_n^i$
   - X_n is integral over $K[y_1, \ldots, y_{n-1}]$ by the Integrality Lemma
   - Therefore each X_i is integral over $K[y_1, \ldots, y_{n-1}]$ (as a sum/difference of integral elements)
4. We can now conclude:
   - $X_1, \ldots, X_n$ are integral over $K[y_1, \ldots, y_{n-1}]$
   - $y_1, \ldots, y_{n-1}$ are integral over $K[z_1, \ldots, z_d]$
   - By transitivity of integrality, $X_1, \ldots, X_n$ are integral over $K[z_1, \ldots, z_d]$
   - Therefore $A = K[X_1, \ldots, X_n]/I$ is integral over $K[z_1, \ldots, z_d]$
5. The elements $z_1, \ldots, z_d$ remain algebraically independent in A because:
   - They are algebraically independent in $K[y_1, \ldots, y_{n-1}]$
   - $K[y_1, \ldots, y_{n-1}]$ embeds in A by the Key Property

Thus we have found algebraically independent elements $z_1, \ldots, z_d$ such that A is integral over $K[z_1, \ldots, z_d]$, completing the proof of Noether's Normalization Lemma. □

Remark 4.7. The proof demonstrates a powerful technique: we first reduce to studying a ring with fewer generators (via the y_i), apply induction to find good elements there (the z_i), and then show these work for our original ring through careful analysis of integrality relations.

Proposition 4.8. *The number d obtained in the construction equals the transcendence degree of A over K.*

Proof. Since A is integral over $K[z_1, \ldots, z_d]$:

$$\operatorname{trdeg}_K A = \operatorname{trdeg}_K K[z_1, \ldots, z_d] = d$$

Any set of algebraically independent elements must have size at most $\operatorname{trdeg}_K A$. Therefore, d is minimal. □

□

Corollary 4.9 (Geometric Interpretation). *For any affine variety $V \subseteq \mathbb{A}^n$, there exists a finite surjective morphism $V \to \mathbb{A}^d$ where $d = \dim V$.*

Proof. Let $A = K[V]$ be the coordinate ring of V. By Noether's Normalization, find $y_1, \ldots, y_d \in A$ with A integral over $K[y_1, \ldots, y_d]$. This gives a finite morphism $V \to \mathbb{A}^d$. Surjectivity follows from the Going-Up theorem. □

Theorem 4.10 (Going-Up). *Let $\phi : R \to S$ be an integral ring homomorphism of commutative rings. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals in R, and let $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_k$ be a chain of prime ideals in S with $k < n$ such that $\phi^{-1}(\mathfrak{q}_i) = \mathfrak{p}_i$ for $i \leq k$. Then the chain in S can be extended to a chain $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_n$ with $\phi^{-1}(\mathfrak{q}_i) = \mathfrak{p}_i$ for all i.*

Example 4.11. Consider the cusp $V(y^2 - x^3) \subseteq \mathbb{A}^2$. The normalization map:

$$t \mapsto (t^2, t^3)$$

exhibits $\mathbb{A}^1$ as the normalization of the cusp. This is a concrete instance of Noether Normalization showing how a singular curve can be parametrized by a line.

5 Hilbert's Nullstellensatz

Definition 5.1 (Radical of an Ideal). The radical of an ideal I in a commutative ring R, denoted $\sqrt{I}$ or $\operatorname{rad}(I)$, is the set

$$\sqrt{I} = \{x \in R : x^n \in I \text{ for some } n > 0\}$$

An ideal I is called radical (or semiprime) if $I = \sqrt{I}$.

Theorem 5.2 (Hilbert's Nullstellensatz). *Let K be an algebraically closed field and let I be an ideal in $K[x_1, \ldots, x_n]$. Then:*

$$\sqrt{I} = I(V(I))$$

where $I(V(I))$ is the ideal of polynomials vanishing on $V(I)$.

Proof. We divide the proof into several stages:

Lemma 5.3 (Weak Nullstellensatz). *If K is algebraically closed and $I \subsetneq K[x_1, \ldots, x_n]$ is a proper ideal, then $V(I) \neq \emptyset$.*

Proof. By Noether's Normalization Lemma (after a linear change of coordinates), there exist elements $y_1, \ldots, y_r$ in $K[x_1, \ldots, x_n]$ such that:

- $K[x_1, \ldots, x_n]/I$ is integral over $K[y_1, \ldots, y_r]$
- $y_1, \ldots, y_r$ are algebraically independent mod I

Since I is proper, $r \geq 1$ (otherwise $K[x_1, \ldots, x_n]/I$ would be integral over K, making it finite-dimensional and therefore 0-dimensional).

Choose any element $a_1 \in K$ and define $\phi_1 : K[y_1] \to K$ by $\phi_1(y_1) = a_1$. For $i = 2, \ldots, r$, we can successively extend ϕ_{i-1} to $\phi_i : K[y_1, \ldots, y_i] \to K$ by choosing any value $a_i \in K$ for y_i. This gives a homomorphism $\phi : K[y_1, \ldots, y_r] \to K$.

By the integrality of $K[x_1, \ldots, x_n]/I$ over $K[y_1, \ldots, y_r]$, for each x_i we have an equation:

$$x_i^{m_i} + c_{i1} x_i^{m_i - 1} + \cdots + c_{im_i} = 0 \mod I$$

where $c_{ij} \in K[y_1, \ldots, y_r]$.

Let $b_{ij} = \phi(c_{ij}) \in K$. Since K is algebraically closed, each equation:

$$X^{m_i} + b_{i1}X^{m_i-1} + \cdots + b_{im_i} = 0$$

has a solution $\alpha_i \in K$.

The point $(\alpha_1, \ldots, \alpha_n) \in K^n$ satisfies all equations in I, hence lies in $V(I)$. $\square$

Lemma 5.4 (Rabinowitsch Trick)**.** *For any ideal $I \subseteq K[x_1, \ldots, x_n]$ and polynomial f, if f vanishes on $V(I)$, then*

$$V(I + \langle 1 - yf \rangle) = \emptyset$$

where y is a new variable.

Proof. Suppose for contradiction that $(a_1, \ldots, a_n, b) \in V(I + \langle 1 - yf \rangle)$. Then $(a_1, \ldots, a_n) \in V(I)$, so $f(a_1, \ldots, a_n) = 0$. But also $1 - bf(a_1, \ldots, a_n) = 0$, implying $1 = 0$, a contradiction. $\square$

Now we prove the main theorem. Let I be an ideal in $K[x_1, \ldots, x_n]$. We show $\sqrt{I} = I(V(I))$.

First, let $f \in \sqrt{I}$. Then there exists $m > 0$ such that $f^m \in I$. For any point $p \in V(I)$, all elements of I vanish at p, so $f^m(p) = 0$. Therefore $f(p) = 0$, showing $f \in I(V(I))$. This proves $\sqrt{I} \subseteq I(V(I))$.

For the reverse inclusion, let $f \in I(V(I))$. Consider the ideal:

$$J = I + \langle 1 - yf \rangle \subseteq K[x_1, \ldots, x_n, y]$$

By the Rabinowitsch Trick, $V(J) = \emptyset$. By the Weak Nullstellensatz, this implies $J = K[x_1, \ldots, x_n, y]$. Therefore $1 \in J$, so there exist $g \in I$ and $h \in K[x_1, \ldots, x_n, y]$ such that:

$$1 = g + h(1 - yf)$$

Multiply this equation by f^m where $m = \deg_y(h) + 1$:

$$f^m = f^m g + f^m h(1 - yf)$$

Consider this equation in $K[x_1, \ldots, x_n, y]$. The right side has degree at most $m-1$ in y. Setting $y = \frac{1}{f}$ (formally) in this equation:

$$f^m = f^m g + 0$$

Therefore $f^m \in I$, showing $f \in \sqrt{I}$. This proves $I(V(I)) \subseteq \sqrt{I}$. $\square$

Corollary 5.5 (Strong Nullstellensatz, Alternative Form)**.** *If K is algebraically closed and I is a proper ideal of $K[x_1, \ldots, x_n]$, then*

$$I = I(V(I)) \iff I \textit{ is radical}$$

Corollary 5.6 (Ideal-Variety Correspondence)**.** *Let K be algebraically closed. There are one-to-one correspondences:*

1. *$\{$Points in $\mathbb{A}^n_K\} \leftrightarrow \{$Maximal ideals in $K[x_1, \ldots, x_n]\}$*
2. *$\{$Irreducible varieties in $\mathbb{A}^n_K\} \leftrightarrow \{$Prime ideals in $K[x_1, \ldots, x_n]\}$*
3. *$\{$Algebraic sets in $\mathbb{A}^n_K\} \leftrightarrow \{$Radical ideals in $K[x_1, \ldots, x_n]\}$*

These correspondences are given by $V \mapsto I(V)$ with inverse $I \mapsto V(I)$.

Corollary 5.7 (Ideal-Variety Dictionary). *For an algebraically closed field K:*

1. $V(I) = \emptyset \iff I = K[x_1, \ldots, x_n]$
2. $V(I) \subseteq V(J) \iff \sqrt{I} \supseteq \sqrt{J}$
3. $V(I \cap J) = V(IJ) = V(I) \cup V(J)$
4. $V(\sum_i I_i) = \bigcap_i V(I_i)$

6 Maximal Spectrum

Definition 6.1 (MaxSpec). For a Noetherian ring R, the maximal spectrum $\mathrm{MaxSpec}(R)$ is the set of all maximal ideals of R equipped with the Zariski topology. The closed sets in this topology are of the form

$$V(I) = \{\mathfrak{m} \in \mathrm{MaxSpec}(R) : I \subseteq \mathfrak{m}\}$$

where I ranges over the ideals of R.

Theorem 6.2. *For a Noetherian ring R, $\mathrm{MaxSpec}(R)$ is a compact space.*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $\mathrm{MaxSpec}(R)$. Each U_α is the complement of some closed set $V(I_\alpha)$:

$$U_\alpha = \mathrm{MaxSpec}(R) \setminus V(I_\alpha)$$

Since these cover $\mathrm{MaxSpec}(R)$:

$$\bigcup_{\alpha \in A} U_\alpha = \mathrm{MaxSpec}(R)$$

Taking complements:

$$\bigcap_{\alpha \in A} V(I_\alpha) = \emptyset$$

This means that no maximal ideal can contain all the I_α. Therefore:

$$\sum_{\alpha \in A} I_\alpha = R$$

In particular, $1 \in \sum_{\alpha \in A} I_\alpha$. Thus there exist elements $r_i \in I_{\alpha_i}$ for finitely many indices $\alpha_1, \ldots, \alpha_n$ such that:

$$1 = r_1 + r_2 + \cdots + r_n$$

This means that:

$$V(I_{\alpha_1}) \cap V(I_{\alpha_2}) \cap \cdots \cap V(I_{\alpha_n}) = \emptyset$$

Taking complements:

$$U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n} = \mathrm{MaxSpec}(R)$$

Therefore, we have found a finite subcover, proving compactness. □

7 Spectrum of a Ring

Definition 7.1 (Spectrum)**.** For a commutative ring R, the spectrum $\operatorname{Spec}(R)$ is the set of all prime ideals of R equipped with the Zariski topology.

Remark 7.2. The commutativity requirement is essential in this definition. For non-commutative rings, the set of prime ideals may not have a suitable topology to form a spectral space. While non-commutative generalizations of the spectrum exist in non-commutative geometry, they are considerably more complex and require different techniques.

Definition 7.3 (Zariski Topology on Spec)**.** The Zariski topology on $\operatorname{Spec}(R)$ of a commutative ring R is defined by taking the closed sets to be of the form

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) : I \subseteq \mathfrak{p}\}$$

where I ranges over the ideals of R.

Proposition 7.4. *For any commutative ring R, $\operatorname{Spec}(R)$ with the Zariski topology satisfies:*

(a) $V(0) = \operatorname{Spec}(R)$ *and* $V(R) = \emptyset$

(b) $V(I_1) \cup V(I_2) = V(I_1 I_2) = V(I_1 \cap I_2)$

(c) $V(\sum_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$

Proof. Parts (a) and (c) follow directly from the definitions.

For (b), note that a prime ideal contains $I_1 I_2$ if and only if it contains either I_1 or I_2 (by the prime property). Also, $V(I_1 \cap I_2) = V(I_1 I_2)$ since for any ideal K, $I_1 I_2 \subseteq K$ if and only if $I_1 \cap I_2 \subseteq K$. □

Theorem 7.5. *For any commutative ring R, $\operatorname{Spec}(R)$ is a spectral space, meaning:*

1. *It is quasi-compact (every open cover has a finite subcover)*
2. *It is sober (every irreducible closed subset is the closure of a unique point)*
3. *The quasi-compact open sets form a basis closed under finite intersections*

Proof. For quasi-compactness: Let $\{U_i\}$ be an open cover of $\operatorname{Spec}(R)$. Each U_i is the complement of $V(I_i)$ for some ideal I_i. Then:

$$\bigcup_i U_i = \operatorname{Spec}(R) \implies \bigcap_i V(I_i) = \emptyset \implies V(\sum_i I_i) = \emptyset$$

This means $\sum_i I_i = R$. Therefore, $1 \in \sum_i I_i$. By finite generation of ideals:

$$1 = r_1 + r_2 + \cdots + r_n$$

where $r_i \in I_i$ for finitely many indices i. These finitely many U_i cover $\operatorname{Spec}(R)$.

For sobriety: Let Z be an irreducible closed subset of $\operatorname{Spec}(R)$. Then $Z = V(I)$ for some ideal I. Let $\mathfrak{p} = \bigcap_{\mathfrak{q} \in Z} \mathfrak{q}$. We claim $\mathfrak{p}$ is prime:

If $xy \in \mathfrak{p}$ but $x, y \notin \mathfrak{p}$, then there exist $\mathfrak{q}_1, \mathfrak{q}_2 \in Z$ with $x \notin \mathfrak{q}_1$ and $y \notin \mathfrak{q}_2$. But then $V(I + (x)) \cup V(I + (y))$ would be a proper decomposition of Z, contradicting irreducibility.

For the quasi-compact open basis: The basic open sets are of the form $D(f) = \operatorname{Spec}(R) \setminus V(f)$ for $f \in R$.

1. $D(f) \cap D(g) = D(fg)$ shows closure under finite intersections
2. Quasi-compactness of $D(f)$ follows from quasi-compactness of $\mathrm{Spec}(R_f)$, where R_f is the localization

□

8 Polynomial Maps and Algebra Homomorphisms

We now establish the fundamental correspondence between polynomial maps of varieties and homomorphisms of their coordinate rings. This provides a key bridge between geometric and algebraic perspectives.

Theorem 8.1 (Contravariant Functor)**.** *Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine varieties over an algebraically closed field k. There is a natural bijection:*

$$\{\textit{Polynomial maps } V \to W\} \leftrightarrow \{k\textit{-algebra homomorphisms } k[W] \to k[V]\}$$

Proof. Given a polynomial map $\phi : V \to W$ with coordinate functions $\phi_1, \ldots, \phi_m \in k[V]$, define

$$\phi^* : k[W] \to k[V]$$

by sending each coordinate function y_i on W to $\phi_i \in k[V]$. This extends uniquely to a k-algebra homomorphism since $k[W]$ is generated by the y_i.

Conversely, given a k-algebra homomorphism $\psi : k[W] \to k[V]$, define $\phi : V \to k^m$ by

$$\phi(p) = (\psi(y_1)(p), \ldots, \psi(y_m)(p))$$

Since ψ preserves the relations defining W, the image of ϕ lies in W.

These constructions are inverse to each other: Starting with ϕ, the composition $\phi^*(y_i)$ is just ϕ_i by definition. Starting with ψ, the composition recovers ψ on generators and hence everywhere. □

Proposition 8.2. *This correspondence is functorial: If $\phi : V \to W$ and $\psi : W \to U$ are polynomial maps, then*

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*$$

This extends naturally to the level of prime and maximal ideals:

Theorem 8.3 (MaxSpec Functor)**.** *Let $\phi : V \to W$ be a polynomial map of affine varieties. The induced homomorphism $\phi^* : k[W] \to k[V]$ gives rise to maps:*

$$\begin{aligned} \phi_{\mathrm{Spec}} : \mathrm{Spec}(k[V]) &\to \mathrm{Spec}(k[W]) \\ \mathfrak{p} &\mapsto (\phi^*)^{-1}(\mathfrak{p}) \end{aligned}$$

and

$$\begin{aligned} \phi_{\mathrm{Max}} : \mathrm{MaxSpec}(k[V]) &\to \mathrm{MaxSpec}(k[W]) \\ \mathfrak{m} &\mapsto (\phi^*)^{-1}(\mathfrak{m}) \end{aligned}$$

Proof. We need to verify:

1. If $\mathfrak{p}$ is prime in $k[V]$, then $(\phi^*)^{-1}(\mathfrak{p})$ is prime in $k[W]$

2. If $\mathfrak{m}$ is maximal in $k[V]$, then $(\phi^*)^{-1}(\mathfrak{m})$ is maximal in $k[W]$

For (1): If $fg \in (\phi^*)^{-1}(\mathfrak{p})$, then $\phi^*(fg) = \phi^*(f)\phi^*(g) \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, either $\phi^*(f) \in \mathfrak{p}$ or $\phi^*(g) \in \mathfrak{p}$, so f or g is in $(\phi^*)^{-1}(\mathfrak{p})$.

For (2): For maximal ideals, we can use the geometric interpretation: Maximal ideals correspond to points, and the preimage of a point under a polynomial map corresponds to the preimage of its maximal ideal. □

Corollary 8.4 (Geometric Interpretation)**.** *Under the identification of points with maximal ideals:*

1. $\phi_{\text{Max}}(\mathfrak{m}_p) = \mathfrak{m}_{\phi(p)}$ *for any point* $p \in V$
2. *The map* $\phi_{\text{Max}} : \text{MaxSpec}(k[V]) \to \text{MaxSpec}(k[W])$ *is the same as* $\phi : V \to W$ *under the identification of varieties with their maximal spectra*

Example 8.5. Consider the inclusion $i : \mathbb{A}^1 \to \mathbb{A}^2$ given by $x \mapsto (x, 0)$. The corresponding algebra homomorphism is

$$i^* : k[x, y] \to k[t], \quad x \mapsto t, \quad y \mapsto 0$$

The induced map on maximal ideals takes

$$\mathfrak{m}_a = (t - a) \mapsto (x - a, y)$$

which is indeed the maximal ideal corresponding to the point $(a, 0)$ in $\mathbb{A}^2$.

This functorial perspective reveals that: 1. Geometric maps correspond naturally to algebraic maps in the opposite direction 2. The correspondence respects composition and algebraic operations 3. The MaxSpec functor provides a bridge between geometric and algebraic viewpoints 4. Points and maximal ideals are fundamentally the same object viewed from different perspectives

Remark 8.6. This correspondence is a key example of the general principle in algebraic geometry that we can study geometric objects through their rings of functions, with geometric maps corresponding to algebraic ones in the opposite direction.

9 Krull Dimension

The notion of dimension in algebraic geometry can be approached in several equivalent ways. The most algebraic approach is through chains of prime ideals:

Definition 9.1 (Chain of Prime Ideals)**.** Let R be a ring. A chain of prime ideals is a sequence

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

where each $\mathfrak{p}_i$ is a prime ideal. We call n the length of the chain.

Definition 9.2 (Height)**.** The height of a prime ideal $\mathfrak{p}$ in a ring R, written $(\mathfrak{p})$, is the supremum of lengths of chains of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$$

In other words, $(\mathfrak{p})$ measures how far $\mathfrak{p}$ is above minimal prime ideals.

Definition 9.3 (Krull Dimension)**.** The Krull dimension of a ring R, written $\dim R$, is the supremum of the lengths of all chains of prime ideals in R. For a variety V over a field k, we define its dimension $\dim V$ to be the Krull dimension of its coordinate ring $k[V]$.

There's another natural way to measure dimension - by counting algebraically independent elements:

Definition 9.4 (Transcendence Degree)**.** For a field extension L/k, the transcendence degree $\mathrm{trdeg}_k(L)$ is the size of any transcendence basis (maximal algebraically independent subset) of L over k. For a k-algebra A, we write $\mathrm{trdeg}_k(A)$ for the transcendence degree of its fraction field over k.

A fundamental result about heights of prime ideals is:

Theorem 9.5 (Principal Ideal Theorem)**.** *Let R be a Noetherian ring and let $x \in R$ be a non-zero-divisor. If $\mathfrak{p}$ is a minimal prime ideal containing (x), then $(\mathfrak{p}) = 1$.*

Proof. Let $\mathfrak{p}$ be minimal over (x). Suppose for contradiction that $(\mathfrak{p}) > 1$. Then we have a chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}$$

with $(x) \subseteq \mathfrak{p}_0$.

In $R/(x)$, this gives a chain

$$\bar{\mathfrak{p}}_0 \subsetneq \bar{\mathfrak{p}}_1 \subsetneq \bar{\mathfrak{p}}$$

Since x is a non-zero-divisor, $\dim R/(x) = \dim R - 1$. But $\bar{\mathfrak{p}}$ is minimal in $R/(x)$ (as $\mathfrak{p}$ was minimal over (x)), contradicting that minimal primes have height 0. □

The remarkable fact is that our different notions of dimension coincide:

Proposition 9.6 (Dimension Formula)**.** *Let k be a field and let A be a finitely generated k-algebra. Then*

$$\dim(A) = \sup_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathrm{trdeg}_k(\mathrm{Frac}(A/\mathfrak{p})).$$

Moreover, we only need to look at minimal prime ideals. If A is an integral domain, this simplifies to:

$$\dim(A) = \mathrm{trdeg}_k(\mathrm{Frac}(A)).$$

Proof. The proof is difficult and will be omitted. It can be found in [David Eisenbud: *Commutative algebra with a view toward algebraic Geometry*, p. 290, Theorem A]. □

Here's a fundamental example:

Corollary 9.7. *The polynomial ring $k[X_1, \ldots, X_n]$ has Krull dimension n.*

Proof. The chain

$$(X_1, \ldots, X_n) \supset (X_1, \ldots, X_{n-1}) \supset \cdots \supset (X_1) \supset (0)$$

shows $\dim k[X_1, \ldots, X_n] \geq n$. The dimension formula then gives equality. □

Theorem 9.8. *If A is an integral domain containing a field k and finitely generated as a k-algebra, then*

$$\mathrm{trdeg}_k(\mathrm{Frac}(A)) = \dim(A).$$

Proof. By Noether's Normalization, A is integral over a polynomial subring $k[x_1, \ldots, x_n]$ where $n = \mathrm{trdeg}_k(\mathrm{Frac}(A))$. Going-up shows $\dim(A) \geq n$, and the dimension formula gives equality. □

10 Affine Schemes

Definition 10.1 (Locally Ringed Space)**.** A locally ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for each point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

Definition 10.2 (Affine Scheme)**.** An affine scheme is a locally ringed space that is isomorphic to $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for some commutative ring R, where $\mathcal{O}_{\mathrm{Spec}(R)}$ is the structure sheaf of R.

Theorem 10.3. *For any commutative ring R, $\mathrm{Spec}(R)$ has a natural structure of a locally ringed space where:*

1. *The structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ assigns to each open set U the ring of regular functions on U*
2. *For each $\mathfrak{p} \in \mathrm{Spec}(R)$, the stalk $\mathcal{O}_{\mathfrak{p}}$ is naturally isomorphic to $R_{\mathfrak{p}}$*

11 Projective Varieties

Definition 11.1 (Projective Space)**.** The projective space $\mathbb{P}^n_k$ over a field k is the set of equivalence classes of $(n+1)$-tuples $(x_0 : \cdots : x_n)$ of elements of k, not all zero, under the equivalence relation

$$(x_0 : \cdots : x_n) \sim (\lambda x_0 : \cdots : \lambda x_n)$$

for any $\lambda \in k^\times$.

Definition 11.2 (Homogeneous Polynomial)**.** A polynomial $f \in k[x_0, \ldots, x_n]$ is homogeneous of degree d if every monomial in f has total degree d.

Definition 11.3 (Projective Variety)**.** A projective variety in $\mathbb{P}^n_K$ is an irreducible algebraic set defined by homogeneous polynomials.

Definition 11.4 (Singular Point)**.** A point p on a variety V is singular if the rank of the Jacobian matrix at p is less than the dimension of V. A variety is smooth if it has no singular points.

Definition 11.5 (Quadratic Form)**.** Let V be a vector space over a field k. A quadratic form on V is a function $Q : V \to k$ such that:

1. $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in k$ and $v \in V$
2. The function $B_Q(v, w) = Q(v + w) - Q(v) - Q(w)$ is bilinear

Given a basis $\{e_1, \ldots, e_n\}$ of V, any quadratic form can be written uniquely as

$$Q(x_1 e_1 + \cdots + x_n e_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

where $a_{ij} = a_{ji}$ are elements of k. The matrix $A = (a_{ij})$ is called the matrix of Q with respect to the given basis.

Remark 11.6. If $\mathrm{char}(k) \neq 2$, then B_Q is symmetric and determines Q via:

$$Q(v) = \frac{1}{2} B_Q(v, v)$$

When $\mathrm{char}(k) = 2$, this relationship breaks down, and quadratic forms require more careful treatment.

Definition 11.7 (Non-degenerate Quadratic Form)**.** A quadratic form Q is called non-degenerate if its matrix A has full rank, or equivalently, if the associated bilinear form B_Q is non-degenerate (i.e., if $B_Q(v, w) = 0$ for all w implies $v = 0$).

Theorem 11.8 (Conic Classification)**.** *Let C be an irreducible conic in $\mathbb{P}^2_k$ where k is algebraically closed. Then:*

1. *If C is smooth, then $C \cong \mathbb{P}^1_k$*
2. *If C is singular, then C consists of two lines meeting at a point*

Proof. Any conic can be written as:

$$ax^2 + by^2 + cz^2 + dxy + eyz + fxz = 0$$

For the smooth case: First, since we assume the conic is smooth, its matrix representation

$$\begin{pmatrix} a & d/2 & f/2 \\ d/2 & b & e/2 \\ f/2 & e/2 & c \end{pmatrix}$$

has rank 3 (non-degenerate quadratic form).

Complete the square in x:

$$(ax + \frac{d}{2}y + \frac{f}{2}z)^2 + (b - \frac{d^2}{4a})y^2 + (c - \frac{f^2}{4a})z^2 + (e - \frac{df}{4a})yz = 0$$

After change of coordinates (using that k is algebraically closed), this becomes:

$$X^2 + Y^2 + Z^2 = 0$$

We can parametrize this by:

$$[\lambda : \mu] \mapsto [\lambda^2 - \mu^2 : 2\lambda\mu : i(\lambda^2 + \mu^2)]$$

giving an isomorphism with $\mathbb{P}^1$.

For the singular case: The quadratic form must be degenerate, so after coordinate change (by Sylvester's law of inertia):

$$X^2 + Y^2 = 0$$

This factors as $(X + iY)(X - iY) = 0$, giving two lines intersecting at $[0 : 0 : 1]$. □

Remark 11.9. This proof uses that the field is algebraically closed in three crucial ways:

1. To diagonalize the quadratic form
2. To ensure the existence of square roots in the parametrization
3. To guarantee that every point on the conic is rational over the field

Definition 11.10 (Veronese Map)**.** The Veronese map $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ where $N = \binom{n+d}{d} - 1$ is given by:

$$\nu_d([x_0 : \cdots : x_n]) = [\text{all monomials of degree } d \text{ in } x_0, \ldots, x_n]$$

Remark 11.11 (Intuition Behind the Veronese Map). The Veronese map provides a way to embed projective space into a higher-dimensional projective space while preserving important geometric properties. Its key features are:

1. It converts equations of degree d in the source to linear equations in the target
2. The image forms a projectively normal variety
3. It provides a natural way to study degree d hypersurfaces in $\mathbb{P}^n$

Example 11.12. For $n = 1, d = 2$ the Veronese map is:

$$\nu_2([x : y]) = [x^2 : xy : y^2]$$

Its image is the conic $V(XZ - Y^2)$ in $\mathbb{P}^2$.

Definition 11.13 (Segre Embedding)**.** The Segre embedding $\sigma : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ is given by:

$$\sigma([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) = [\ldots : x_i y_j : \ldots]$$

Remark 11.14 (Intuition Behind the Segre Embedding). The Segre embedding provides a way to understand the product of projective spaces as a projective variety. Its significance includes:

1. It realizes the categorical product in algebraic geometry
2. Its image represents separable tensors in the projectivization of a tensor product
3. It provides a model for studying bilinear forms and their geometry

Example 11.15. For $m = n = 1$, the Segre embedding is:

$$\sigma([x_0 : x_1], [y_0 : y_1]) = [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1]$$

Its image is the quadric surface $V(XW - YZ) \subseteq \mathbb{P}^3$.

Proposition 11.16 (Properties of Segre Embedding)**.** *The image of $\sigma : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ is cut out by the quadrics:*

$$X_{i,j}X_{k,l} = X_{i,l}X_{k,j}$$

where $X_{i,j}$ represents the coordinate corresponding to $x_i y_j$.

Proof. These equations clearly hold on the image since:

$$(x_i y_j)(x_k y_l) = (x_i x_k)(y_j y_l) = (x_i y_l)(x_k y_j)$$

Conversely, given a point satisfying these equations, we can recover the preimage by fixing any non-zero coordinate and solving for the remaining coordinates using the relations. □

12 Polynomial Maps Between Varieties

Definition 12.1 (Regular Map)**.** A regular map between affine varieties V and W is a function $\phi : V \to W$ that can be written locally as a ratio of polynomials where the denominator doesn't vanish. More precisely, for each $p \in V$, there is an open neighborhood U of p and polynomials $f_1, \ldots, f_n, g$ such that:

$$\phi(x) = \left(\frac{f_1(x)}{g(x)}, \ldots, \frac{f_n(x)}{g(x)}\right)$$

for all $x \in U$ where $g(x) \neq 0$.

Example 12.2 (Projection Map)**.** Consider the hyperbola $V = V(xy-1) \subseteq \mathbb{A}^2$ and the punctured line $W = \mathbb{A}^1 \setminus \{0\}$. The projection map $\pi : V \to W$ given by

$$\pi(x, y) = x$$

is both a polynomial map and an isomorphism.

Proof. First, we verify that $\pi(V) \subseteq W$: If $(a, b) \in V$, then $ab = 1$, so $a \neq 0$ and thus $\pi(a, b) = a \in \mathbb{A}^1 \setminus \{0\}$.

The inverse map $\psi : W \to V$ given by $\psi(x) = (x, \frac{1}{x})$ is regular since x is invertible on W. The composition in either direction gives the identity, proving π is an isomorphism. □

Example 12.3 (Normalization of Cusp)**.** Let $V = V(y^2 - x^3) \subseteq \mathbb{A}^2$ be the cuspidal cubic. The map $\phi : \mathbb{A}^1 \to V$ given by:

$$\phi(t) = (t^2, t^3)$$

provides a normalization of V.

Proposition 12.4. *The map ϕ is:*

1. *Well-defined (image lies in V)*
2. *Surjective*
3. *Injective except at $t = 0$*
4. *A normalization of V*

Proof. For well-definedness: For any $t \in \mathbb{A}^1$:

$$(\phi(t))_2^2 - (\phi(t))_1^3 = (t^3)^2 - (t^2)^3 = t^6 - t^6 = 0$$

For surjectivity: Let $(a, b) \in V$. If $(a, b) = (0, 0)$, then $(0, 0) = \phi(0)$. Otherwise, $t = \sqrt[3]{b}$ works since $b^2 = a^3$.

For injectivity except at 0: If $\phi(s) = \phi(t)$, then $s^2 = t^2$ and $s^3 = t^3$. If $s \neq 0$, this implies $s = t$. At $t = 0$, we have the singular point.

For normalization: $\mathbb{A}^1$ is normal, and ϕ induces a finite birational map. The result follows from the universal property of normalization. □

Example 12.5 (Twisted Cubic)**.** The twisted cubic curve is the image of $\mathbb{P}^1$ under the map:

$$\phi([s : t]) = [s^3 : s^2t : st^2 : t^3]$$

Proposition 12.6. *The image of ϕ is the intersection of the quadrics:*

$$XZ = Y^2$$
$$YW = Z^2$$
$$XW = YZ$$

Proof. First, we verify these equations hold on the image by direct substitution: Let $X = s^3$, $Y = s^2t$, $Z = st^2$, $W = t^3$. Then:

$$XZ = s^3(st^2) = s^4t^2 = (s^2t)^2 = Y^2$$
$$YW = (s^2t)(t^3) = s^2t^4 = (st^2)^2 = Z^2$$
$$XW = s^3t^3 = (s^2t)(st^2) = YZ$$

Conversely, given a point $[a : b : c : d]$ satisfying these equations:

- If $a \neq 0$, set $s = 1$ and solve for t
- If $d \neq 0$, set $t = 1$ and solve for s

This shows every point satisfying the equations lies in the image. □

13 Applications and Exercises

Example 13.1 (Computing Dimension)**.** Let's compute the Krull dimension of $k[x, y, z]/(xy-z^2)$:

Proof. First, note that $xy - z^2$ is irreducible (as it is quadratic in z and cannot be factored), so $k[x, y, z]/(xy - z^2)$ is an integral domain and (0) is prime.

By the Principal Ideal Theorem, any minimal prime over $(xy - z^2)$ has height 1. Since $xy - z^2$ is irreducible, $(xy - z^2)$ is itself prime of height 1.

Every maximal ideal containing $(xy - z^2)$ has height 2 (it cannot be more as $\dim k[x, y, z] = 3$ and we've used height 1 for $(xy - z^2)$).

Therefore, $\dim k[x, y, z]/(xy - z^2) = 2$. □

The following exercises explore fundamental properties of algebraic geometry. Complete solutions are provided after the exercises.

Show that for any field k, $\dim k[x_1, \ldots, x_n] = n$.

Prove that if R is a PID, then $\mathrm{Spec}(R)$ has dimension 1.

Show that the projective closure of the curve $y = x^2$ in $\mathbb{A}^2$ has exactly one point at infinity.

14 Solutions to Exercises

[Solution to Exercise 1] We prove $\dim k[x_1, \ldots, x_n] = n$ in two steps.

First, we show $\dim k[x_1, \ldots, x_n] \geq n$: Consider the chain of prime ideals:

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \ldots, x_n)$$

This chain has length n, showing the dimension is at least n.

For the reverse inequality, let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_m$ be any chain of prime ideals.

By Noether's Normalization Lemma, $k[x_1, \ldots, x_n]/\mathfrak{p}_0$ has transcendence degree at most n over k. Each successive quotient reduces transcendence degree by at least 1, so $m \leq n$.

Therefore $\dim k[x_1, \ldots, x_n] = n$.

[Solution to Exercise 2] Let R be a PID. We prove $\operatorname{Spec}(R)$ has dimension 1.

First, let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a chain of prime ideals in R.

Since R is a domain, (0) is prime and must be $\mathfrak{p}_0$.

In a PID, any non-zero prime ideal is maximal. Indeed, if (p) is prime and $(p) \subsetneq I$ for some ideal I, then $I = R$ since every ideal is principal.

Therefore, $\mathfrak{p}_1$ must be maximal, so $n = 1$.

To show $\dim R \geq 1$, observe that for any prime element $p \in R$, we have the chain:

$$(0) \subsetneq (p)$$

Therefore $\dim R = 1$.

[Solution to Exercise 3] To find the projective closure of $y = x^2$, we:

1. Homogenize the equation: Replace x with $\frac{X}{Z}$ and y with $\frac{Y}{Z}$ to get:

$$\frac{Y}{Z} = \left(\frac{X}{Z}\right)^2$$

Multiply through by Z^2:

$$YZ = X^2$$

2. At infinity (set $Z = 0$):

$$0 = X^2$$

Therefore $X = 0$, and Y can be any non-zero value.

3. After normalization, this gives exactly one point $[0 : 1 : 0]$ at infinity.

4. For uniqueness:

- Any point at infinity must satisfy $Z = 0$
- This forces $X = 0$ by the equation
- Since not all coordinates can be zero, $Y \neq 0$
- All points $[0 : a : 0]$ for $a \neq 0$ are equivalent to $[0 : 1 : 0]$ in projective space

Therefore, there is exactly one point at infinity.